Homework 0

Due: Wednesday, August 31, at 11am.

This is a "callibration" homework (for both of us). **This homework, unlike the others, must be done individually.** It is worth a ε-fraction of your grade, where $\varepsilon > 0$ is a insignificantly small. Please give this assignment a serious effort but don't let it ruin your day. **Pro-tip:** cheating on this assignment makes the class harder.

1. **(Matrices.)**

   a. Given matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$, show $\text{tr}(AB) = \text{tr}(BA)$.

   b. Given a square symmetric matrix $C \in \mathbb{R}^{n\times n}$, show $\text{tr}(C) = \sum_i \lambda_i$, where λ_i is the i^{th} eigenvalue of C.

2. **(Cauchy-Schwarz.)** The Cauchy-Schwarz inequality says that every pair of vectors $a, b \in \mathbb{R}^n$ satisfy $|a^\top b| \leq \|a\|\|b\|$ (and there are more general versions).

   a. Given reals $(c_1, \ldots, c_k)$, show
   $$\sum_{i=1}^{k} \frac{c_i}{i} \leq \sqrt{2}\sqrt{\sum_{i=1}^{k} c_i^2}.$$
   (**Note:** if you are ambitious, the $\sqrt{2}$ can be replaced with $\pi/\sqrt{6}$.)

   b. Prove the following inequality which is similar to Cauchy-Schwarz: $|a^\top b| \leq \|a\|_1\|b\|_\infty$, where $\|a\|_1 = \sum_i |a_i|$ and $\|b\|_\infty = \max_i |b_i|$.

3. **(Geometric puzzle.)** Consider the following game: you are given a set of points in the plane, and you must either establish that for every subset there exists an axis-aligned rectangle which contains just the points in that subset (and avoids the points outside the subset), or you must prove that this is impossible.

   a. First find a set of 4 points such that the game is impossible; namely, find a set of 4 points in the plane and a subset of them and then prove that that any axis-aligned rectangle containing the subset must also contain some other points. **(Feel free to include a scan of a hand-drawn figure.)**

   b. Next find a set of 4 points in the plane where the game is successful: show that every subset can be exactly captured by some axis-aligned rectangle.

   c. Now prove that for *any* (distinct) set of 5 points, the game must fail.

   Congratulations! You have proved the VC-dimension of axis-aligned rectangles is 4. (I realize this makes the question google-able; please see the "pro-tip" above.)

(**Hints.** (1b) Real square symmetric matrices always have real eigenvalues and eigenvectors. (3c) Take *any* five (distinct) points, and consider their bounding box.)